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Note on the Use of Supplementary Curves in Isogonal Transformation.

BY ROLLIN A. HARRIS.

1. The main object of this note is to show how the problem of representing one plane conformably upon another; using any real function* of the variable, may be made to depend upon the problem of constructing supplementary curves from given tracings of the corresponding principal curves.

2. It will be convenient to begin by determining the conditions for a monogenic function of the variable $x + kw$.

\bar{X} , W and x , w are real quantities; $k \equiv ij = \sqrt{-1}$ where $i, j = \sqrt{-1}$. Suppose \bar{X} , W to depend upon x , w ; then the ratio of the change in $\bar{X} + kW$ to the change in $x + kw$ will be

$$\frac{d\bar{X} + kdW}{dx + kdw}. \quad (1)$$

This ratio will be independent of the direction in which $d(x + kw)$ is taken, i. e. independent of the variable quantities dx , dw , if

$$\frac{\partial \bar{X}}{\partial x} = \frac{\partial W}{\partial w}, \quad \frac{\partial \bar{X}}{\partial w} = \frac{\partial W}{\partial x}. \quad (2)$$

It may be noted here that unless equations (2) are satisfied the quotient (1) will, in general, become infinite when the variable $x + kw$ is taken along one of the two systems of lines

$$w = \pm x + \text{constant}; \quad (3)$$

but when equations (2) are satisfied this difficulty disappears, for the quotient is then equal to $\frac{\partial \bar{X}}{\partial x} + k \frac{\partial W}{\partial x}$.

*The term "real function" will be applied to any monogenic function, $X + iY$, of $x + iy$ such that X remains unaltered when y is replaced by $-y$, and Y becomes $-Y$ by the same substitution.

3. Let ϕ denote any real function, and suppose $\phi(x + kw)$ to be developable by Taylor's theorem in powers of kw within certain regions of convergence; then

$$\phi(x) + \frac{w^2}{2!}\phi''(x) + \dots = \cosh\left(w\frac{\partial}{\partial x}\right) \cdot \phi(x) = \bar{X}, \quad (4)$$

$$w\phi'(x) + \frac{w^3}{3!}\phi'''(x) + \dots = \sinh\left(w\frac{\partial}{\partial x}\right) \cdot \phi(x) = W, \quad (5)$$

which satisfy the conditions for a monogenic function.

4. *Isogonality in general.* Let X, Y and x, y denote, in this paragraph, real or imaginary quantities. Suppose X, Y to so depend upon x, y that

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}, \quad (6)$$

then at the intersections* of any two curves (real or imaginary),

$$y = f_1(x), \quad y = f_2(x), \quad (7)$$

we have

$$\tan^{-1}\left(\frac{dY}{dX}\right)_1 - \tan^{-1}\left(\frac{dY}{dX}\right)_2 \equiv \tan^{-1}\left(\frac{dy}{dx}\right)_1 - \tan^{-1}\left(\frac{dy}{dx}\right)_2, \quad (8)$$

where the subscript indicates the curve taken and multiples of π are disregarded. If an angle be measured in the contrary sense, as compared with the others, its sign in (8) must be altered. If a measurement start in advance of the initial direction by a certain amount, this amount must be added to one or the other of the members of (8).

Since X and Y each depend upon x, y , the relation $y = f(x)$ leads to a relation between X and Y . The curve, or portion of curve, (real or imaginary) representing this relation and corresponding to $y = f(x)$, is a complete image of $y = f(x)$.

If, now, we define

$$\tan^{-1}\left(\frac{dY}{dX}\right), \text{ or } \tan^{-1}\left(\frac{dy}{dx}\right),$$

as the angle which a tangent to a curve makes with the X -axis, or the x -axis, it follows that the difference between two such angles belonging to the intersecting

*It is assumed that at these points, $\frac{\partial X}{\partial x}$ and $\frac{\partial Y}{\partial x}$ do not both vanish or both become infinite.

images equals the difference between the two corresponding angles of the intersecting paths (7), regardless of the reality of the curves or angles. In this sense the angles are preserved.

5. In like manner, if the variable point (x, w) be taken along any two intersecting curves, and if \bar{X} , W so depend upon x, w that

$$\frac{\partial \bar{X}}{\partial x} = \frac{\partial W}{\partial w}, \quad \frac{\partial \bar{X}}{\partial w} = \frac{\partial W}{\partial x},$$

we have, in general,

$$\tanh^{-1}\left(\frac{dW}{d\bar{X}}\right)_1 - \tanh^{-1}\left(\frac{dW}{d\bar{X}}\right)_2 \equiv \tanh^{-1}\left(\frac{dw}{dx}\right)_1 - \tanh^{-1}\left(\frac{dw}{dx}\right)_2, \quad (9)$$

where multiples of $j\pi$ are disregarded. If one of the hyperbolic angles;

$$\tanh^{-1}\left(\frac{dW}{d\bar{X}}\right)_1,$$

be reckoned from the W -axis, instead of from the \bar{X} -axis, and in the contrary sense, this relation becomes

$$\tanh^{-1}\left(\frac{d\bar{X}}{dW}\right)_1 - \tanh^{-1}\left(\frac{d\bar{X}}{dW}\right)_2 \equiv \tanh^{-1}\left(\frac{dw}{dx}\right)_1 - \tanh^{-1}\left(\frac{dw}{dx}\right)_2, \quad (10)$$

since

$$\tanh^{-1} \infty = \frac{j\pi}{2}, \quad \tanh^{-1} \frac{1}{a} = \frac{j\pi}{2} - \tanh^{-1} a,$$

and multiples of $j\pi$ are disregarded.

This form is useful when a real hyperbolic tangent is numerically greater than unity.

6. If $y, =f(x)$, be such a function of x that its values are real or purely imaginary for all real values of x , the entire curve $y=f(x)$ can always be drawn as a real curve in one or both of the two planes xy, xw where jw replaces y .

If both planes are required for the curve, the part in the one will be said to be supplementary* to the part in the other.

* Strictly speaking, the part in the one plane would have to be rotated into the plane of the other part before it could be called supplementary. For convenience this rotation may be left unperformed.

the plane xy ,
 If $y = f(x)$ lie in the plane xw ,
 the planes xy and xw ,

the plane XY .

its complete image, by any real function ϕ , lies in the plane $\bar{X}W$.

the planes XY and $\bar{X}W$.

7. *Measurement of hyperbolic angles.* As a real angle whose tangent is known may be represented by twice the area of a circular sector, so a real hyperbolic angle whose hyperbolic tangent is known may be represented by twice the area of an hyperbolic sector. The radius of the circle is unity; the hyperbola is rectangular, having its semi-axis equal to unity. The area ($s/2$) of the sector is connected with the true angle at the center of the hyperbola by the equation

$$\tan \theta = \tanh s.$$

This enables one to construct an hyperbolic protractor whose divisions represent equal areas.

8. If the variable $x + kw$ describe any two real intersecting paths in the xw -plane, and $\bar{X} + kW$ describe two real intersecting curves in the $\bar{X}W$ -plane, the angles of intersection will be preserved—all angles being measured with an hyperbolic protractor, and in accordance with §§4, 5.

All straight lines cutting the x -axis at a true angle of $\pm 45^\circ$ transform into straight lines cutting the \bar{X} -axis at a true angle of $\pm 45^\circ$. If the xw -plane be divided into rectangles by the pair of systems $w = \pm x + \text{constant}$, the $\bar{X}W$ -plane (or at least a certain portion of it) will also be divided into rectangles by the corresponding systems $W = \pm \bar{X} + \text{constant}$.

9. *Quasi images.* The result of transforming any curve or point in the xw -plane to the $\bar{X}W$ -plane, by means of a monogenic function, may be called the quasi image of that curve or point.

The last remark in §8 enables one to construct mechanically the quasi image of any given path in the xw -plane, as soon as he has computed the real values of the transforming function corresponding to the real values of the variable x . For, the pairs of systems there mentioned determine all corresponding points in the two planes.

If \bar{X} and W are each symmetric in x, w , then the quasi image of $x = f(w)$ has the same equation as has that of $w = f(x)$. If \bar{X} is the same function of x, w as W is of w, x , then the equation of the quasi image of $x = f(w)$ may be obtained from that of $w = f(x)$ by interchanging \bar{X} and W .

10. Application to the construction of curves supplementary to the true images of certain paths symmetric with respect to the x -axis.

If $y = f(x)$ denote any real curve symmetric about the x -axis, its true image, whose equation is got by eliminating x and y from

$$y = f(x) \\ X + iY = \phi(x + iy),$$

is symmetric about the X -axis, ϕ being a real function.

Now suppose f to be such a function that for certain real values of x, y becomes a pure imaginary. Replacing y by jw (§6), we obtain a real locus, $jw = f(x)$, supplementary to the real part of $y = f(x)$. The quasi image of this is got by eliminating x and jw from the equations

$$jw = f(x) \\ \bar{X} + ijW = \phi(x + jw)$$

since $k = ij$. But this resultant is the same function of \bar{X}, jW as the former resultant was of X, Y ; hence the following:

*The quasi image of the supplement of a given path is supplementary to the true image of the same path.**

(In representing a region of the xy -plane conformably upon the XY -plane, it is usually best to select very simple systems of curves for paths of the variable. Having thus determined corresponding points in the two planes, the image of any given path can be drawn at once.)

11. *Special case.* The quasi image of $x = \alpha$ is supplementary to the true image of $x = \alpha$.

If the function $\phi(x)$ have a focus† upon the X -axis, this point is a real focus of the true image of the system $x = \text{constant}$ ‡. For, the quasi image of the line

* The \bar{X} -axis is now supposed to coincide with the X -axis.

† i. e. a point in whose immediate vicinity the transformed elements are indefinitely small in comparison with their former magnitude.

‡ The nature of ϕ , and the particular focus chosen, when there are several such foci, determine the limits between which this constant may be taken.

$x = \alpha$ (because it crosses the X -axis and passes through opposite angles of the elementary rectangles into which the XW -plane is divided by the systems $W = \pm X + \text{constant}$) must touch the two lines in the XW -plane passing through the focus of the function making angles of $\pm 45^\circ$ with the X -axis. Now these two lines, when referred to the XY -plane, still pass through the focus of the function; they also pass through the circular points at infinity, and are imaginary lines tangent to the true image of $x = \alpha$, which is supplementary to the quasi image of $x = \alpha$.

Example. If $\phi \equiv \text{sine}$, the quasi image of the system $x = \text{constant}$ is a system of ellipses inscribed in a square having two of its opposite corners at the points $X = \pm 1, W = 0$ which are the foci of $\phi(x)$. Consequently the true image of the system $x = \text{constant}$ is a system of confocal hyperbolas having the points $X = \pm 1, Y = 0$ for foci. To construct the system of ellipses mechanically (§9), lay off upon the X -axis the values of $\sin x$ as x varies uniformly from $-\pi/2$ to $+\pi/2$. Draw two systems of straight lines cutting the X -axis at angles of $\pm 45^\circ$, and join the opposite corners of the elementary rectangles thus formed; the curves are the required quasi image of the system $x = \text{constant}$ by the function ϕ .

12. Suppose

$$X + kW = (x + kw)^2,$$

and let $x + kw$ move over the straight lines

$$x = \alpha, \quad w = mx;$$

$X + kW$ will describe a parabola, and a straight line passing through the focus of the supplementary parabola.

\therefore Given any parabola of the second order whose axis of symmetry is made to coincide with the X -axis, and any secant line passing through the focus of the supplementary curve; let tangents be drawn at the points of intersection: the one will be inclined as much to the X -axis as the other will be to the W -axis,—the angles being measured in opposite directions (§§4, 5, 7).

If

$$X + kW = (x + kw)^n,$$

where n is any integer greater than unity, and $x + kw$ move as before, the above statement holds true if the words “any parabola of the second order” be replaced by “certain curves of the n th order.”

If

$$X + kW = (x + kw)^{\frac{1}{2}}$$

and $x + kw$ move over the straight lines

$$x = \alpha, \quad w = \delta,$$

$X + kW$ will describe a circle and a rectangular hyperbola. If they intersect, let tangents be drawn to the two curves at a point of intersection. The tangent to the circle cuts the one asymptote at the same angle as the tangent to the hyperbola does the other.

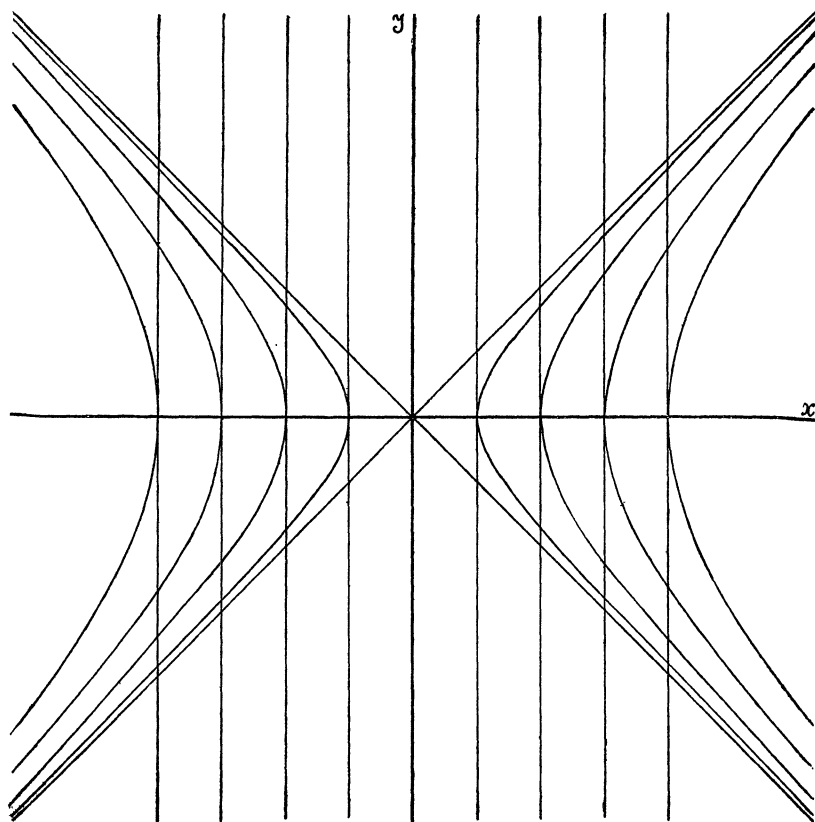


FIG. 1.

13. The accompanying figures illustrate the simple case where

$$X + iY = (x + iy)^2,$$

and where the variable is taken over the systems $x = \text{constant}$ and

$x^2 - y^2 = (\text{constant})^2$. Of course $X + iY$ will describe a system of confocal parabolas and a system of straight lines parallel to the Y -axis.

Figs. 1 and 2 may remain the same for various transforming functions; Fig. 3 is obtained from Fig. 2 by mechanical construction (§9), $\phi(x)$ having been laid off upon the X -axis.

The problem which this note leads up to, is that of passing in general from Fig. 3 to Fig. 4; i. e. from certain given curves to their supplementaries.

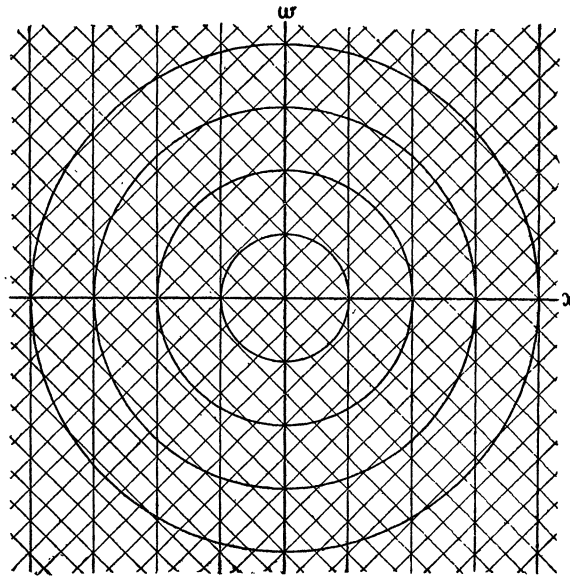


FIG. 2.

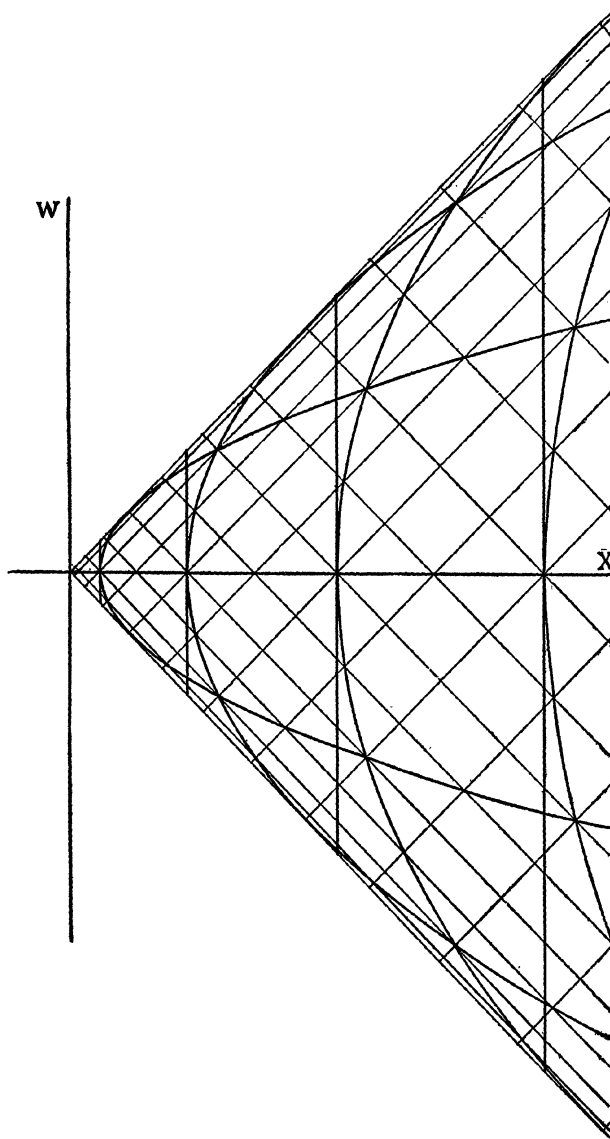


FIG. 3.

Scale $\frac{1}{4}$ that of Figs. 1 and 2.)

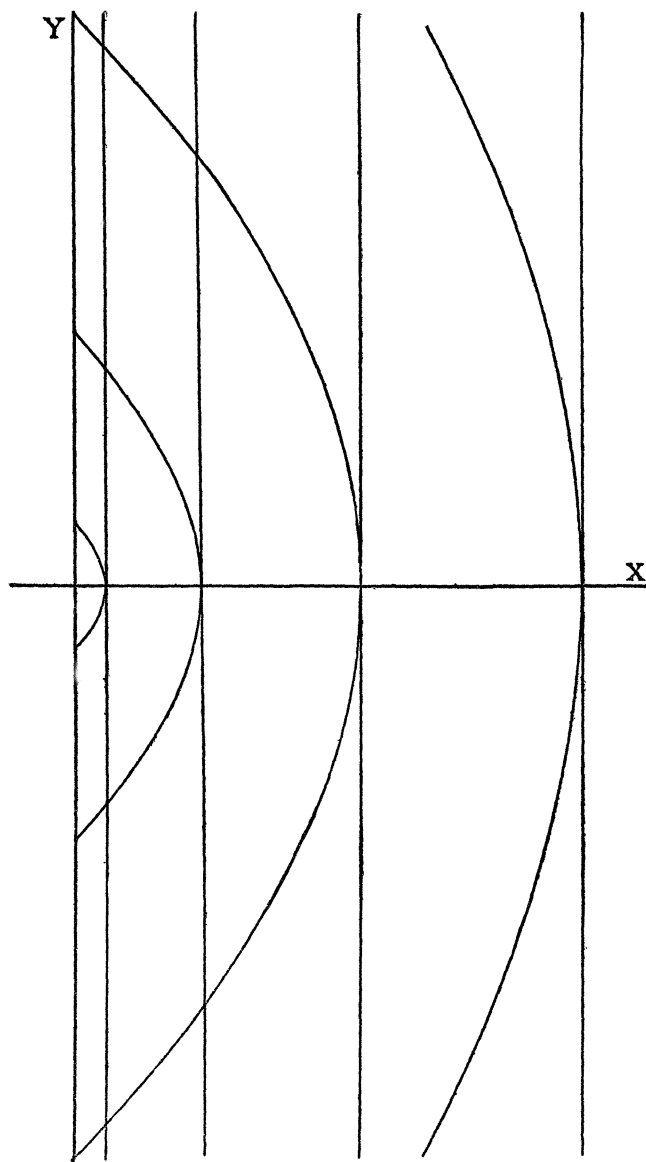


FIG. 4.

(Scale $\frac{1}{2}$ that of Figs. 1 and 2.)